MATHEMATICS

1. Ratio of coefficients three consecutive terms in proportion of $(1 + \alpha)^n$ is 1 : 5 : 20 then find value of n

Ans. 29

Sol. ${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = a : b : c$

Then $n + 1 = \dfrac{(a+b)(b+c)}{b^2 - b^2 - ac}$

$\Rightarrow n + 1 = = \dfrac{6 \times 25}{25 - 20} = 30$

n = 29

2. Find the coefficient of the term in the expansion of $\left(3x^2 - \dfrac{1}{2x^5}\right)^7$ which is independent of x.

Ans. 1275.75

Sol. $T_{r+1} = {}^7C_r (3x^2)^{7-r}\left(-\dfrac{1}{2x^5}\right)^r$

$= {}^7C_r 3^{7-r}\left(-\dfrac{1}{2}\right)^r (x^{14-2r-5r})$

$x^0 \Rightarrow r = 2$

coefficient $= {}^7C_2 3^5\left(-\dfrac{1}{2}\right)^2$

$= 21 \times 3^5 \times \dfrac{1}{4} = \dfrac{5103}{4}$

= 1275.75 **Ans.**

3. If $P = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $Q = P^TAP$, then $PQ^{2023} P^T$ is

Ans. $\begin{bmatrix} 1 & 2023 \\ 0 & 1 \end{bmatrix}$

Sol. $PP^T = I$ and $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$Q^2 = P^T\ A\ P\ P^T\ A\ P$

$= P^T\ A^2\ P$

$Q^3 = P^T\ A^3\ P$

$\Rightarrow P\ Q^{2023}\ P^T = P \cdot P^T\ A^{2023}\ P^T\ P$

$= A^{2023}$

$= \begin{bmatrix} 1 & 2023 \\ 0 & 1 \end{bmatrix}$

4. There are 8 boys and 6 girls. Find the number of arrangements if they are sitting in a circular arrangement such that no two girls are together.

Ans. $(7! \times {}^8C_6 \times 6!)$

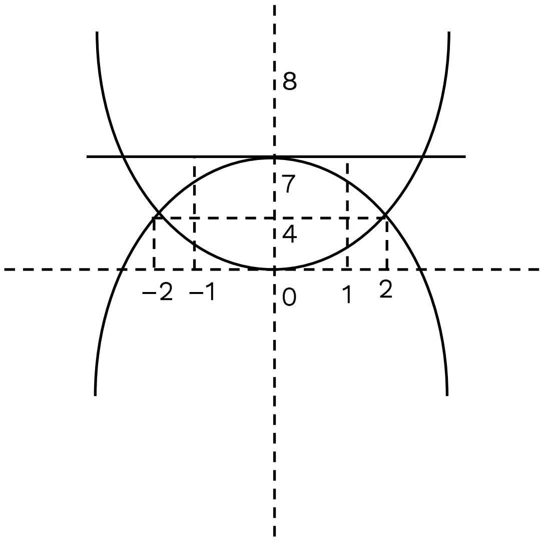
Sol. **Step 1** $\Rightarrow$ Let the boys first occupy the chairs in circle $\Rightarrow 7!$

Step 2 $\Rightarrow$ Gap between boys will be occupied by girls $\Rightarrow [7! \times {}^8C_6 \times 6!]$

5. Find area bounded by $y \le 7, y \ge x^2$ & $y \le 8 - x^2$.

Ans. 20

Sol.



Area = $2 \times \frac{2}{3} \times 16 - \frac{2}{3} \times 2$

$= \frac{2}{3}(32 - 2) = 2 \times \frac{30}{2} = 20$

6. Find the number of words formed from the word “INDEPENDENCE”, such that vowels are together.

Ans. $\frac{8!}{3!2!} \times \frac{5!}{4!}$

Sol. $I^1, N^3, D^2, E^4, C^1, P^1$

$\boxed{I, E, E, E, E} + N^3, D^2, C^1, P^1$

$\frac{8!}{3!2!} \times \frac{5!}{4!}$

7. Dot product of two vectors is 12 and cross product is $4\hat{i} + 6\hat{j} + 8\hat{k}$, find product of modulus of vectors.

Ans. $\sqrt{260}$

Sol. Let $\vec{a}$ and $\vec{b}$ be two vectors

$|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 = 144 + 16 + 36 + 64$

$= 144 + 116 = 260$

$|\vec{a}||\vec{b}| = \sqrt{260}$

8. Find the summation of $\sum_{i=1}^{n} P$, where (P = 1 + 2 + 3 + ... k)

Ans. $\frac{k(k+1)(k+2)}{6}$

Sol. $P = 1 + 2 + 3 + \ldots\ldots.k = \frac{k(k+1)}{2}$

$$\sum_{1=1}^{n} P = \sum_{1=1}^{n} \frac{k^2+k}{2} = \frac{1}{2}\left[\frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2}\right]$$

$$= \frac{1}{4}k(k+1)\left[\frac{2k+1+3}{3}\right]$$

$$= \frac{1}{4} \times \frac{k(k+1) \times 2(k+2)}{3}$$

$$= \frac{k(k+1)(k+2)}{6}$$

9. Find the negation of $[(p \Rightarrow q) \to (q \to p)]$

Ans. $\sim p \wedge q$

Sol. $\sim[(\sim p \vee q) \to (\sim q \vee p)]$

$\Rightarrow \sim[\sim(\sim p \vee q) \vee (\sim q \vee p)]$

$\Rightarrow \sim[(p \wedge \sim q) \vee (\sim q \vee p)]$

$= \sim[p \vee \sim q]$

$= \sim p \wedge q$

10. Given that 66! is completely divisible by 3^n find the maximum integral value of n.

Ans. 31

Sol. $\frac{66!}{3^n} = 22 + 7 + 2 = \left[\frac{66}{3}\right] + \left[\frac{66}{3^2}\right] + \left[\frac{66}{3^3}\right]$

$\Rightarrow 22 + 7 + 2 = 31$

11. Let $A[a_{ij}]_{3\times3}, |adjA| = 16$ then find the value of $|adj(adj(adj(2A)))|$.

Ans. 2^{40}

Sol. $|adj\ adj|adj\ 2a||^{(3-1)^3} = |2A|^8$

$= (2^3|A|)^8 = 2^{24}|A|^8$

$|adj\ A| = 16 \Rightarrow |A|^2 = 16 \Rightarrow |A| = 4$

$= 2^{24} \times 4^8 = 2^{40}$

12. Shortest distance between lines $\frac{x-5}{4} = \frac{y-3}{6} = \frac{z-2}{4}$ and $\frac{x-3}{7} = \frac{y-2}{5} = \frac{z-9}{6}$ is

Ans. $\frac{95}{\sqrt{189}}$

Sol. $\vec{n_1}\times\vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 4 \\ 7 & 5 & 6 \end{vmatrix}$

$= \hat{i}(16) - \hat{j}(-4) + \hat{k}(-22)$

$= 2(8\hat{i} + 2\hat{j} - 11\hat{k})$

$= \vec{AC} = 2\hat{i} + \hat{j} - 7\hat{k}$

$SD = \left|\dfrac{\vec{AC}.(\vec{n_1}\times\vec{n_2})}{(\vec{n_1}\times\vec{n_2})}\right| = \dfrac{16+2+77}{\sqrt{189}}$

$= \dfrac{95}{\sqrt{189}}$

13. Integrate $\displaystyle\int \frac{(x+1)}{x(1+xe^x)^2}\,dx$

Ans. $\dfrac{1}{1+x.e^x} + \ell n\left(\dfrac{x.e^x}{1+x.e^x}\right) + C$

Sol. $\displaystyle\int \frac{e^x(x+1)}{x\cdot e^x(1+xe^x)^2}dx$

$1 + x \cdot e^x = t$

$(xe^x + e^x)dx = dt$

$\Rightarrow \displaystyle\int \frac{dt}{(t-1)t^2} = \int \frac{1}{t}\left(\frac{-1}{t} + \frac{1}{t-1}\right)dt$

$= \displaystyle\int \left(\frac{-1}{t^2} + \frac{1}{t(t-1)}\right)dt$

$= \dfrac{1}{t} + \ell n(t-1) - \ell nt + C$

$= \dfrac{1}{1+x\cdot e^x} + \ell n\left(\dfrac{(1+x\cdot e^x - 1)}{(1+x\cdot e^x)}\right) + C$

$= \dfrac{1}{1+x\cdot e^x} + \ell n\left(\dfrac{x\cdot e^x}{1+x\cdot e^x}\right) + C$ Ans.

14. If the mean and variance of the given eight numbers x, y, 10, 12, 6, 12, 4, 8 be 9 and 9.25 and x > y then find 3x − 2y = ?

Ans. 25

Sol. $\dfrac{x+y+52}{8} = 9$

$9.25 = \dfrac{x^2+y^2+100+144+36+144+16+64}{8} - (9)^2$

$9.25 = \dfrac{x^2+y^2+504}{8} - 81$

$74 = x^2 + y^2 + 504 - 648$

$x^2 + y^2 = 722 - 504 = 218$

$x + y = 20$ & $x^2 + y^2 = 218$

$x = 13$, $y = 7$

$3x - 2y = 39 - 14 = 25$

15. There are three factories A, B & C and production in these factories are 20%, 30% and 50% respectively and defected products are 2%, 3% & 7% respectively. One item is selected randomly & found to be defective then find the probability that it is produced by machine C

Ans. $\frac{35}{48}$

Sol. $\text{Probability} = \dfrac{\frac{50}{100}\times\frac{7}{100}}{\frac{20}{100}\times\frac{2}{100}+\frac{30}{100}\times\frac{3}{100}+\frac{50}{100}\times\frac{7}{100}}$

$= \dfrac{350}{40+90+350}$

$= \dfrac{35}{48}$

16. If $T_n = \dfrac{n^3}{n^4+147}$ maximum value of T_n

Ans. $T_5 > T_4$

Sol. $f(x) = \dfrac{x^3}{x^4+147}$

$\Rightarrow f'(x) = \dfrac{3x^2\left(x^4+147\right)-4x^3.x^3}{\left(x^4+147\right)^2}$

$\Rightarrow \dfrac{441x^2-x^6}{\left(x^4+147\right)^2} = \dfrac{x^2}{\left(x^2+147\right)^2}\left(441-x^4\right)$

Hence $f'(x) = 0$ at $x^4 = 441 \Rightarrow x^2 = 21$

$\Rightarrow$ Maximum occurs at $x = 4$ or $x = 5$

$T_4 = \dfrac{64}{256+147},\ T_5 = \dfrac{125}{625+147}$

$T_4 = \dfrac{64}{403},\ T_5 = \dfrac{125}{772} \quad \Rightarrow \quad T_5 > T_4$

17. $f(x) = x + \int_0^1 t(x+t)f(t)dt$, find $\dfrac{23}{2}$ $f(0)$ =?

Ans. 9

Sol. $f(x) = x + x\int_0^1 t\ f(t)\ dt + \int_0^1 t^2\ f(t)\ dt$

Let $\int_0^1 tf(t)dt = p,\ \int_0^1 t^2 f(t) = q$

So, $f(x) = (p + 1)\, x + q$

Now $\int_0^1 t\left\{(p+1)t+q\right\}\ dt = p$ & $\int_0^1 t^2\ \left\{(p+1)t+q\right\}dt = q$

$\Rightarrow (p+1)\dfrac{1}{3}+\dfrac{q}{2} = p$ & $\dfrac{p+1}{4}+\dfrac{q}{3} = q$

$\Rightarrow 2p+2+3q = 6p$ & $3p+3+4q = 12q$

$\Rightarrow 3\left(4p-3q=2\right)$ & $\left(3p-8q=-3\right)4$

$\Rightarrow q = \frac{18}{23}$

$p = \frac{25}{23}$

$f(x) = (p+1)x + q$

$= \left(\frac{25}{23} + 1\right)x + \frac{18}{23}$

$f(x) = \frac{48}{23}x + \frac{18}{23}$

18. The given function is even or odd. $f(x) = \frac{(1+2^x)^7}{2^x}$ is:

(1) Even
(2) Odd
(3) Neither even non odd
(4) None of these

Ans. (3)

Sol. $f(-x) = \frac{(1+2^{-x})^7}{2^{-x}}$

$f(-x) = \left(\frac{2^x + 1}{2^x}\right)^7 \times 2^x$

$f(-x) = \frac{2^x + 1}{2^{6x}}$

So neither even non odd

19. Find the solution of differential equation $(y - 2\log_e x)dx = -2x\log x\,dy$

Ans. $2y\log x = (\log x)^2 + C$

Sol. $(y - 2\log_e x)dx = -2x\log x\ dy$

$\Rightarrow \frac{dy}{dx} = \frac{y}{-2x\log x} + \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{-2x\log x}\right) = \frac{1}{x}$

$\Rightarrow dy + P(x).dx = Q(x).dx$ line D.E.

$\text{I.F.} = e^{\int P(x)dx} = e^{\int \frac{1}{x}\cdot\frac{1}{\log x}.dx} = e^{\int \frac{1}{t}dt}$

$\text{I.F.} = e^{\ell nt} = t = \log x$

Y.I.F. $\int Q(x)\ \text{I.F.}\ dx$

$\Rightarrow y\log x = \int \frac{1}{x}\log x\ dx\quad t = \log x$

$\Rightarrow y\log x = \frac{(\log x)^2}{2} + C$

$\Rightarrow 2y\log x = (\log x)^2 + C$

20. A has 5 elements and 'B' has 2 elements. Find the number of subsets of A × B such that the number of elements is more than or equal to 3 and less than 6 is:

Ans. 582

Sol. $n(A \times B) = 5 \times 2 = 10$

The number of subsets $= {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5$

$= \dfrac{10 \times 8 \times 9}{3 \times 2} + \dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} + \dfrac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2}$

$= 120 + 210 + 36 \times 7$

$= 120 + 210 + 252 = 582$

21. Let the circum-centre of triangle formed by the lines $3x - 4y = -5$, $4x + 3y = 53$ and $x - y = 1$ is (α, β). Then the value of $(\alpha - \beta)^2 + (\alpha + \beta)$ is

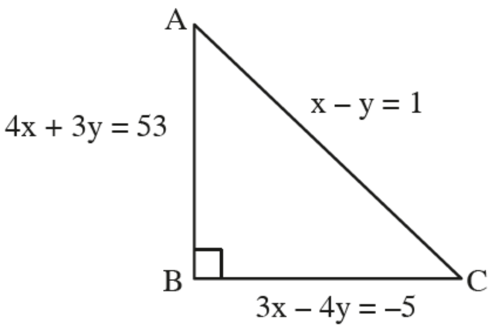
(1) 16 (2) 17

(3) 18 (4) 19

Ans. (2)

Sol. $\because$ ABC is right angled

$\therefore$ Circumcentre of ΔABC is mid-point of AC solving



$4x + 3y = 53$ & $x - y = 1$, we get $A \equiv (8, 7)$

Similarly, $C \equiv (9, 8)$

$\therefore\ \alpha = \dfrac{17}{2}$ and $\beta = \dfrac{15}{2}$

$\Rightarrow (\alpha - \beta)^2 + (\alpha + \beta) = 1 + 16 = 17$

22. If roots of $x^3 + bx + c = 0$ are α, β, γ and $\beta\gamma = -1 = \alpha$ then value of $b^3 + c^3 - \alpha^3 + 6\beta^3 + 6\gamma^3$ is

(1) 16 (2) 21 (3) 19 (4) $\dfrac{168}{55}$

Ans. **(1)**

Sol. $\alpha + \beta + \gamma = 0 \Rightarrow \beta + \gamma = 1$

$\alpha\beta + \beta\gamma + \gamma\alpha = b \Rightarrow -1 + \alpha(\beta + \gamma) = b \Rightarrow b = -2$

$\alpha\beta\gamma = -c \Rightarrow c = -(-1)(-1) \Rightarrow c = -1$

$\beta + \gamma = 1,\ \beta^3 + \gamma^3 + 3\beta\gamma(\beta + \gamma) = 1,\ \beta^3 + \gamma^3 + 3(-1)(1) = 1,\ \beta^3 + \gamma^3 = 4$

So, $b^3 + c^3 - \alpha^3 + 6\beta^3 + 6\gamma^3$ is

$-8 - 1 + 1 + 6(4) = 16$

23. The value of $\lim_{x\to 0} \frac{1-\cos^2 3x}{\cos^3 4x} \cdot \frac{\sin^3 4x}{(\ell n(1+2x))^5}$ is

(1) 18 (2) 16 (3) 9 (4) 27

Ans. **(1)**

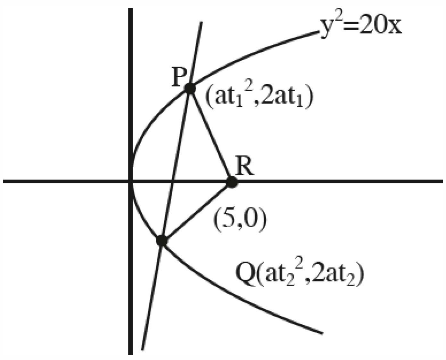
Sol. $$\lim_{x\to 0} \frac{(1-\cos 3x)(1+\cos 3x)(3x)^2 \ \sin^3 4x (4x)^3}{\cos^3 4x \cdot (3x)^2 \quad (4x)^3 \left(\frac{\lim(1+2x)}{2x} \cdot 2x\right)^5}$$

$$\lim_{x\to 0} \frac{\frac{1}{2} \cdot 2 \cdot 9x^2 \cdot 64 \cdot x^3}{32 \cdot x^5} = 18$$

24. If line $y = mx + c$ cuts the parabola $y^2 = 20x$ at P & Q and R is focus of parabola. If centroid of ΔPQR is (10, 10) & $c - m = 6$, then find $(PQ)^2$

Ans. **(325)**

Sol. $\frac{at_1^2 + at_2^2 + 5}{3} = 10,\ \frac{2at_1 + 2at_2 + 0}{3} = 10$



$t_1^2 + t_2^2 = 5,\ t_1 + t_2 = 3$

Now $PQ^2 = (at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2 = 325$